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Spin-up of a source-sink driven flow in a two-layer rotating fluid

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SUMMARY

In this paper the linearized spin-up process of a two-layer fluid in a rotating annulus is examined. The flow is induced by a source and a sink at the inner boundary of the annulus. The spin-up is controlled by the Ekman-suction velocity as well as the moving interface. On the assumption of vanishing small internal and external Froude numbers, the vorticity in each layer is a function of time only and can be expressed in terms of the hypergeometric functions. The components of the velocity can readily be deduced in terms of the vorticity. Some numerical results are given to illustrate the spin-up process.

1. Introduction

In the present paper a time-dependent, source-sink driven flow in a rotating annulus will be investigated. The source and the sink are located at the inner boundary of the annulus, and are located within two immiscible fluids of different density and viscosity. In order to draw attention to the effects of a moving interface and a doubly connected region, the physical configuration is chosen as simple as possible.

The classical spin-up problem of a two-layer fluid in a rotating cylinder was considered by Pedlosky [1], and more generally by Berman, Bradford and Lundgren [2]. For the case of a one-layer fluid, a time-dependent, source-sink flow in a rotating annulus was studied by Barcilon [3], while a sink-driven flow with a free surface was studied by Kranenburg [4]. A related problem was also considered by Kuo and Veronis [5], and a steady source-sink driven flow in a two-layer fluid within a rotating annulus was investigated by Mæland [6].

The fundamental equation which governs the linearized spin-up process is the equation for the vertical component of relative vorticity in each layer. The vorticity is induced by the moving interface and the Ekman-suction velocity from the boundary layers at the interface and the rigid bottom of the rotating annulus. The upper surface is not exposed to any (air) stress. On the assumption that the Ekman-layers are quasi-steady, i.e. that the boundary layers can be considered as essentially steady after a time of order $O(\Omega^{-1})$, where Ω is the Coriolis frequency, we can use the general expressions for the (quasi-steady) Ekman-layer suction velocity at the interface as derived in [6]. It is important to note that we do not account for the early states of the spin-up process when this approximation is used, so in our terminology the initial values need not be zero relative motion in each layer, cf. Greenspan [7].

In the present paper the governing equations are solved under the assumption of vanishing small internal and external Froude numbers. The consequences of this approximation is that the relative vorticity in each layer is a function of time only. Moreover, it implies that the upper free surface and the interface are level surfaces. On the other hand, with this approximation, we are able to solve the resulting equations in terms of well-known functions of mathematical physics (hypergeometric function). The main conclusions of this paper are concerned with the interior (geostrophic), axisymmetric motion.

At the initial point of time, $O(\Omega^{-1})$, the vorticity and the azimuthal velocity are equal to zero in each layer. There is a weak radial and vertical motion in each layer as a consequence of the quasi-steady approximation. As the time increases, the relative vorticity increases/decreases in the upper/lower layer as a consequence of the conservation of the potential vorticity. As soon as the vorticity becomes non-zero, the Ekman-suction will modify this situation and viscous effects will determine the time evolution. The difference in the azimuthal velocities, which will increase from zero, implies that the boundary layer transports also increase from zero. Subsequently, the transport of fluid will be confined to boundary layers, although there will be a weak Ekman-suction into the boundary layer at the bottom of the annulus. The time evolution of the vorticity cannot be described in simple terms, since there are too many (dimensionless) parameters which determine the behavior. The actual solution of the problem necessitates numerical techniques in order to compute the relative vorticity. We present some results in order to illustrate the spin-up process. Finally, the velocity components can readily be deduced in terms of the vorticity.

2. Formulation of the problem

The region between two co-axial cylinders of inner and outer radius a and b is filled with two immiscible fluids of different density and viscosity, which form two layers of thickness h_1 and h_2 (see Figure 1). The cylinders are rotating with a constant angular velocity about the axis of symmetry, which coincides with the vertical direction. We use cylindrical co-ordinates (r, θ, z) fixed in the rotating frame of reference. We also use subscripts $(\ )_1$ and $(\ )_2$ to denote the upper and lower fluid, respectively. The flow is induced by a source of density ρ_1 and viscosity μ_1 at the inner cylinder, and a sink of fluid with density ρ_2 and viscosity μ_2 also at the inner cylinder. The positions of the source and the sink are close to the upper free surface and to the lower bottom, respectively. The strength of the source and the sink is defined in terms of the fluid volume emitted/absorbed in unit time (m^3/s). The interface is assumed to move in the vertical direction with a constant velocity q (m/s), caused by the equal strength of the source and the sink, so that the total volume is conserved (incompressible fluids).

The boundary conditions are no-slip at all rigid boundaries. At the interface we must have continuity of velocity and viscous stress, in addition to the kinematic condition $w = dh/dt$. At the upper free surface we have the condition of no (air) stress.

We will denote the partial derivatives with respect to r, z and t (time) by subscripts $(\ )_r$, $(\ )_z$

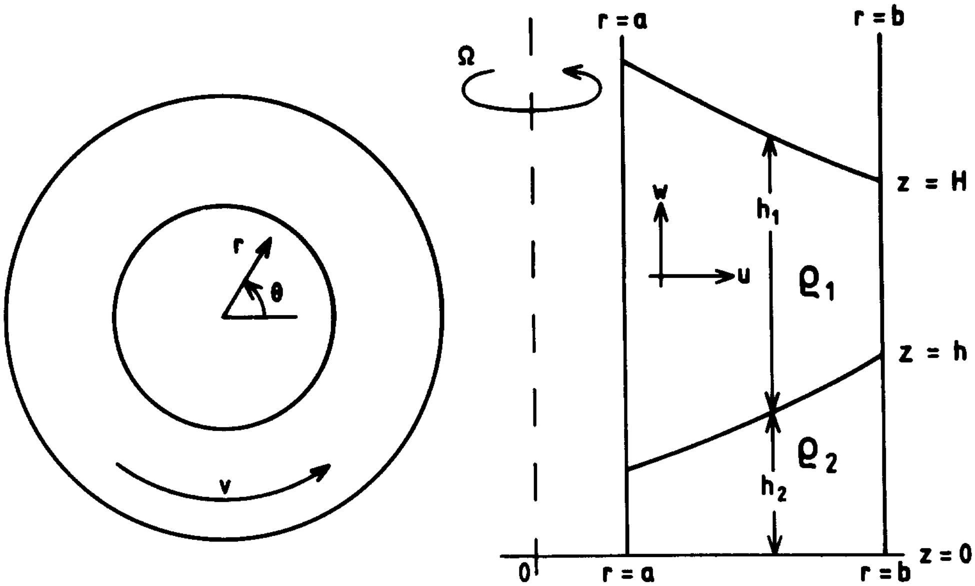

Figure 1. Schematic sketch of the physical configuration (horizontal and vertical cross sections), and the definitions of some physical variables.

and $(\ )_t$, respectively. We use capital letters to denote the interior values of the velocity $\mathbf{U} = (U, V, W)$, and we ask for axisymmetric solutions, i.e. $\partial/\partial\theta \equiv 0$. Away from the viscous boundary layers (the interior regions), the governing equation for the linearized spin-up problem is the vertical component of the vorticity equation

$$Z_t = 2\Omega W_z, \tag{1a}$$

where the relative vorticity (Z) is defined by the relation

$$Z = \frac{1}{r}(rV)_r. \tag{1b}$$

For the interior motion we also have, cf. Greenspan [7],

$$U_z = V_z = W_{zz} = 0 \Rightarrow Z_z = 0, \tag{1c}$$

so the vertical velocity is a linear function of the vertical co-ordinate (z). The vertical velocity is induced by the Ekman layer suction. On the assumption of quasi-steady flow, i.e. that the Ekman-layers become fully developed within a time of $O(\Omega^{-1})$ and then change only slowly, see Greenspan [7], we can use the well-known formulae for the Ekman-suction velocity to obtain W_z in each layer. A detailed derivation was given by Mæland [6], and the results are given here by

$$h_1 W_{1z} = +q - \frac{1}{2}\frac{bE_1^{1/2}}{1+k}(Z_1 - Z_2), \tag{2a}$$

$$h_2 W_{2z} = -q + \frac{1}{2}\frac{bE_2^{1/2}}{1+k}(kZ_1 - (1+2k)Z_2), \tag{2b}$$

where we have denoted the ratio of the coefficients of viscosities by $k^2 = \rho_1\mu_1/\rho_2\mu_2$, and the Ekman numbers in each fluid by $E = \mu/(\rho\Omega b^2)$. The Ekman numbers will be assumed to be small as compared to unity. We will also denote the ratio of the densities by $m = \rho_1/\rho_2$.

The linearized vorticity equations can then be written

$$h_1 Z_{1t} = q(+2\Omega - C_1 Z_1 + C_1 Z_2), \tag{3a}$$

$$h_2 Z_{2t} = q(-2\Omega + kC_2 Z_1 - (1+2k)C_2 Z_2), \tag{3b}$$

where we have introduced the constants C_1 and C_2 defined by

$$C_1 = \frac{b\Omega E_1^{1/2}}{q(1+k)}, \qquad C_2 = \frac{b\Omega E_2^{1/2}}{q(1+k)}. \tag{4a,b}$$

We emphasize that this set of equations presupposes that the Froude number is small. We do not account for the vorticity induced by the radial flow across constant-depth contours, nor the time-dependent deviation from the moving interface. This approximation implies that the thickness of each layer is a function of time only. In fact, we have the relations $h_1 = h_1(t) = H_1 + qt$ and $h_2 = h_2(t) = H_2 - qt$, where H_1 and H_2 are the initial thickness of the two layers, with $H_1 + H_2 = H =$ constant. By inspection of equations (3a,b) we conclude that the relative vorticity is a function of time only, and it is clear that this greatly simplifies the analysis.

Let us first find the particular solutions of equations (3a,b), which are time-independent, Z_{1s} and Z_{2s}, respectively. We obtain

$$(1+k)C_1 C_2 Z_{1s} = 2\Omega((1+2k)C_2 - C_1), \tag{5a}$$

$$(1+k)C_1 C_2 Z_{2s} = 2\Omega(kC_2 - C_1). \tag{5b}$$

In order to facilitate the actual algebra in the present paper, we will use the relation $m^2 E_1 = k^2 E_2$. From the definition of C_1 and C_2, we thus have $mC_1 = kC_2$. We will also define a Rossby number (Ro) in terms of the velocity q. However, since the interior (azimuthal) velocity is generally larger by a multiplicative factor $E^{-1/2}$ than the mass flux producing it, see Mæland [6], we define the Rossby number by

$$\mathrm{Ro}^{-1} = \Omega\frac{b}{q}E_1^{1/2} = (1+k)C_1. \tag{6}$$

The solutions of equations (5a,b) can then be represented by

$$Z_{1s} = 2\Omega \mathrm{Ro} \frac{k}{m}\left(\frac{m}{k}(1+2k)-1\right), \tag{7a}$$

$$Z_{2s} = 2\Omega \mathrm{Ro} \frac{k}{m}(m-1). \tag{7b}$$

We note that since $m \leqslant 1$, $Z_{2s} \leqslant 0$ when $q > 0$ (and vice versa). On the other hand, Z_{1s} can assume both positive and negative values depending on the actual values of k and m. For most practical situations when $1/2 \leqslant m \leqslant 1$, however, we have $Z_{1s} > 0$ when $q > 0$.

3. The vorticity equations

In this section we will give the general solution of the time-dependent vorticity equations (3a,b). Since the thickness of the layers increases/decreases at a constant rate, the coefficients in these equations are not constant. The present problem is thus different from the classical spin-up problem of a two-layer fluid as considered in [1] and [2], where the mean thickness of each layer was constant. The present problem can be solved by a Taylor series expansion in powers of t, viz.

$$Z_1(t) = Z_1(0) + Z_{1t}(0)t + \ldots = +2\Omega qt/H_1 + O(t^2), \tag{8a}$$

$$Z_2(t) = Z_2(0) + Z_{2t}(0)t + \ldots = -2\Omega qt/H_2 + O(t^2), \tag{8b}$$

so the vorticity of the upper layer increases when $q > 0$, since the thickness increases, and vice versa in the lower layer. This is in agreement with the conservation of potential vorticity which can be written $(2\Omega + Z)/h =$ constant. Since the vorticities are zero at time $t = 0$, we obtain

$$2\Omega + Z_1 = 2\Omega h_1/H_1 \Rightarrow Z_1 = 2\Omega(h_1 - H_1)/H_1 = +2\Omega qt/H_1, \tag{9a}$$

$$2\Omega + Z_2 = 2\Omega h_2/H_2 \Rightarrow Z_2 = 2\Omega(h_2 - H_2)/H_2 = -2\Omega qt/H_2. \tag{9b}$$

However, this evolution cannot be true for all time, since the terms representing the viscous dissipation in the vorticity equations are not zero in general. In order to solve the problem, let us introduce the dimensionless variable T defined by

$$h_2 = H_2 - qt = (H_1 + H_2)T, \tag{10a}$$

which gives

$$h_1 = H_1 + qt = (H_1 + H_2)(1 - T), \tag{10b}$$

and finally

$$(H_1 + H_2)\frac{d}{dt} = -q\frac{d}{dT}. \tag{10c}$$

In this way we obtain the following set of equations, where we use the non-dimensional vorticity $\check{Z}$ defined by $Z = 2\Omega\check{Z}$

$$(1-T)\check{Z}_{1T} = -1 + C_1\check{Z}_1 - C_1\check{Z}_2, \tag{11a}$$

$$T\check{Z}_{2T} = +1 - kC_2\check{Z}_1 + (1+2k)C_2\check{Z}_2. \tag{11b}$$

We note that

$$t = 0 \Rightarrow T = T_0 = H_2/(H_1 + H_2), \qquad 0 < T_0 < 1, \tag{12a}$$

and the thickness of the upper and lower layer is zero when

$$h_1 = 0 \Rightarrow t = -H_1/q \Rightarrow T = 1, \qquad (q < 0), \tag{12b}$$

$$h_2 = 0 \Rightarrow t = +H_2/q \Rightarrow T = 0, \qquad (q > 0). \tag{12c}$$

The interval $0 \leqslant T \leqslant T_0$ thus applies to the situation $q > 0$, or $h_2 \to 0$ when $T \to 0$, while the interval $T_0 \leqslant T \leqslant 1$ applies to the situation $q < 0$, i.e. $h_1 \to 0$ when $T \to 1$. Note that only the interval $0 \leqslant T \leqslant 1$ has physical significance.

It is possible to eliminate either $\check{Z}_1$ or $\check{Z}_2$ from equations (11a,b), and we will draw the attention to the homogeneous problem, since a particular solution is given by equations (7a,b). After some simple algebra we obtain an equation of the form

$$T(1-T)\check{Z}_{1TT} + (\gamma - (\alpha + \beta + 1)T)\check{Z}_{1T} - \alpha\beta\check{Z}_1 = 0, \tag{13a}$$

which is the hypergeometric equation, cf. Abramowitz and Stegun [8]. The vorticity in the lower layer can then be deduced from equation (11a)

$$\check{Z}_2 = \check{Z}_1 - (1-T)\check{Z}_{1T}/C_1. \tag{13b}$$

The coefficients α, β and γ are given here by the relations

$$\gamma = -(1+2k)C_2, \tag{14a}$$

$$\alpha + \beta = C_1 - (1+2k)C_2, \tag{14b}$$

$$\alpha\beta = -(1+k)C_1C_2. \tag{14c}$$

The actual values of α and β are readily deduced, viz.

$$\left.\begin{matrix} 2\alpha \\ 2\beta \end{matrix}\right\} = C_1 - (1+2k)C_2 \pm \sqrt{C_1^2 + 2C_1C_2 + (1+2k)^2C_2^2}. \tag{15a,b}$$

The hypergeometric equation has the independent solutions

$$F(\alpha,\beta;\gamma;T), \qquad T^{1-\gamma}F(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; T), \tag{16a,b}$$

provided that γ is not a positive or negative integer, Abramowitz and Stegun [8]. We also note that, in order to compute $\check{Z}_2$, we may use the relation

$$\frac{d}{dT}F(\alpha,\beta;\gamma;T) = \frac{\alpha\beta}{\gamma}F(\alpha+1,\beta+1;\gamma+1;T). \tag{16c}$$

The general solution of the vorticity equations (11a,b) is

$$\check{Z}_1 = AX_1(T) + BY_1(T) + \check{Z}_{1s}, \tag{17a}$$
$$\check{Z}_2 = AX_2(T) + BY_2(T) + \check{Z}_{2s}, \tag{17b}$$

where A and B are two constants of integration. The functions X_1, X_2, Y_1 and Y_2 are given by

$$X_1(T) = F(\alpha,\beta;\gamma;T), \tag{18a}$$
$$X_2(T) = X_1(T) - D_1(1-T)F(\alpha+1,\beta+1;\gamma+1;T), \tag{18b}$$
$$Y_1(T) = T^{1-\gamma}F(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; T), \tag{18c}$$
$$Y_2(T) = Y_1(T) - (1-T)T^{-\gamma}\{D_2TF(2+\alpha-\gamma, 2+\beta-\gamma; 3-\gamma; T) + C_1^{-1}(1-\gamma)F(1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; T)\}, \tag{18d}$$

where the constants D_1 and D_2 are given by

$$D_1 = (1+k)/(1+2k), \qquad D_2 = (1+\alpha-\gamma)(1+\beta-\gamma)/(2-\gamma)C_1. \tag{19a,b}$$

The coefficients A and B must be detemined from the initial condition $\check{Z}_1(T_0) = \check{Z}_2(T_0) = 0$. We then obtain the linear system

$$AX_1(T_0) + BY_1(T_0) = -\check{Z}_{1s}, \tag{20a}$$
$$AX_2(T_0) + BY_2(T_0) = -\check{Z}_{2s}. \tag{20b}$$

The actual calculations of A and B must certainly be done by numerical methods, since the hypergeometric function is not tabulated. This will be done in the next section, where results of the numerical computations will be presented.

In closing this section we will draw some attention to the limit $T \to 0$. We are aware that this limit is of no physical significance, since the boundary layer approach then cannot be justified. The same argument applies to the limit $T \to 1$. Nevertheless, if we investigate this limit more closely, we obtain

$$X_1(0) = 1, \qquad X_2(0) = k/(1+2k), \tag{21a,b}$$

$$Y_1(0) = 0, \qquad Y_2(0) = 0, \tag{21c,d}$$

and it follows that

$$\check{Z}_1(0) = \check{Z}_{1s} + A, \tag{22a}$$

$$\check{Z}_2(0) = \check{Z}_{2s} + kA/(1+2k). \tag{22b}$$

This result deserves some comment, since the steady-state solutions $\check{Z}_{1s}$ and $\check{Z}_{2s}$ are not exactly the limits of $\check{Z}_1(T)$ and $\check{Z}_2(T)$ as $T \to 0$ (unless $A \equiv 0$). We note that the corresponding limit in a one-layer sink flow, as studied by Kranenburg [4], behaves in a more reasonable manner. The discrepancies may be attributed to the method of solution, viz. a time scale which is not the physical time. Since we treat the Ekman-layers as quasi-steady, somé unexpected phenomena may very well occur.

We will also draw some attention to a problem not mentioned so far. In our calculations it was presupposed that the parameter γ given by equation (14a) should be neither a positive nor a negative integer. When $q > 0$, this parameter is negative (and vice versa), and if it happens to be an integer, $Y_1(T)$ is still a valid solution, but $X_1(T)$ must be replaced by a solution which involves a logarithmic term. The singular behavior of that function as $T \to 0$ is not in accordance with the results presented so far. We admit that we cannot give a physical explanation of this phenomenon at this moment. We anticipate that a similar phenomenon occurs when $q < 0$, or when $T \to 1$.

4. Numerical results

In this section we will present some results of the numerical computations. The first task is to construct a function routine which calculates the value of the hypergeometric function, since this function was not available by our computer (UNIVAC 1100/82). This was done by using the definition of the hypergeometric function [8]. We also construct a subroutine which computes the values of the functions $X_1(T)$, $X_2(T)$, $Y_1(T)$ and $Y_2(T)$ defined by equation (18). The two constants of integration A and B must be determined from the linear system (20a,b). We are then able to compute the relative vorticities $\check{Z}_1$ and $\check{Z}_2$ as given by equations (17a,b). We have the set of non-dimensional parameters Ro, k, m and H_1/H_2, which must be given appropriate values. We note that the Rossby number (Ro) may be positive ($q > 0$) or negative ($q < 0$). The initial value T_0 is determined by the ratio H_1/H_2, since $T_0 = (1 + H_1/H_2)^{-1}$.

A lot of numerical experiments have been performed, but we will not overburden the present paper with too many examples. We will depict some results which we believe represent the characteristic features of the spin-up process. We will fix the values of Ro = 0.04 and $T_0 =$ 0.5, but vary the parameters m and k. In Figure 2 we depict the time evolution of the relative vorticities $\check{Z}_1(T)$ and $\check{Z}_2(T)$ when $m = 0.5$ and $k = 1.0$. The vorticity first grows according to

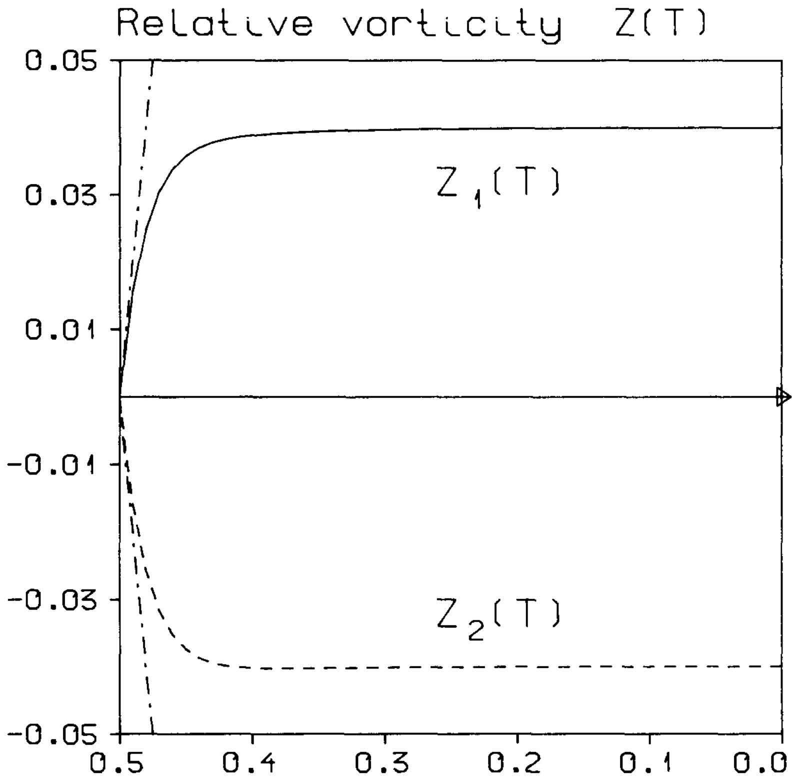

Figure 2. Computer output of the (nondimensional) vorticity $\check{Z}$ as a function of the (nondimensional) time T. Solid curve: $\check{Z}_1$, dashed curve: $\check{Z}_2$, initial value $T_0 = 0.5$. The two straight (dashed) lines represent the vorticities according to inviscid theory. The parameter values are Ro = 0.04, $m = 0.5$ and $k = 1.0$.

inviscid theory (conservation of potential vorticity), but very soon viscous effects invalidate this evolution. The final (steady) states are given here by equations (7a,b):

$$\check{Z}_{1s} = \mathrm{Ro}\,\frac{k}{m}\left(\frac{m}{k}(2k+1)-1\right) = +\mathrm{Ro} = +0.04, \tag{23a}$$

$$\check{Z}_{2s} = \mathrm{Ro}\,\frac{k}{m}(m-1) = -\mathrm{Ro} = -0.04. \tag{23b}$$

We conclude that the two layers are spun up at time $T \approx 0.4$ for this set of parameters.

In Figure 3 we depict the results when $m = 0.25$ and $k = 0.5$. The final (steady) states are given here by $\check{Z}_{1s} = 0$ and $\check{Z}_{2s} = -0.06$, respectively. The response in this case is quite different, since the vorticity in the upper layer first increases and then decreases to its final value. The time needed to reach the final values are somewhat longer than in the previous case, viz. $T \approx 0.3$.

In Figure 4 we depict the results when $m = 0.9$ and $k = 3.0$. The final (steady) states are given here by $\check{Z}_{1s} = 0.1466$ and $\check{Z}_{2s} = -0.0133$. If we compare this case with the previous

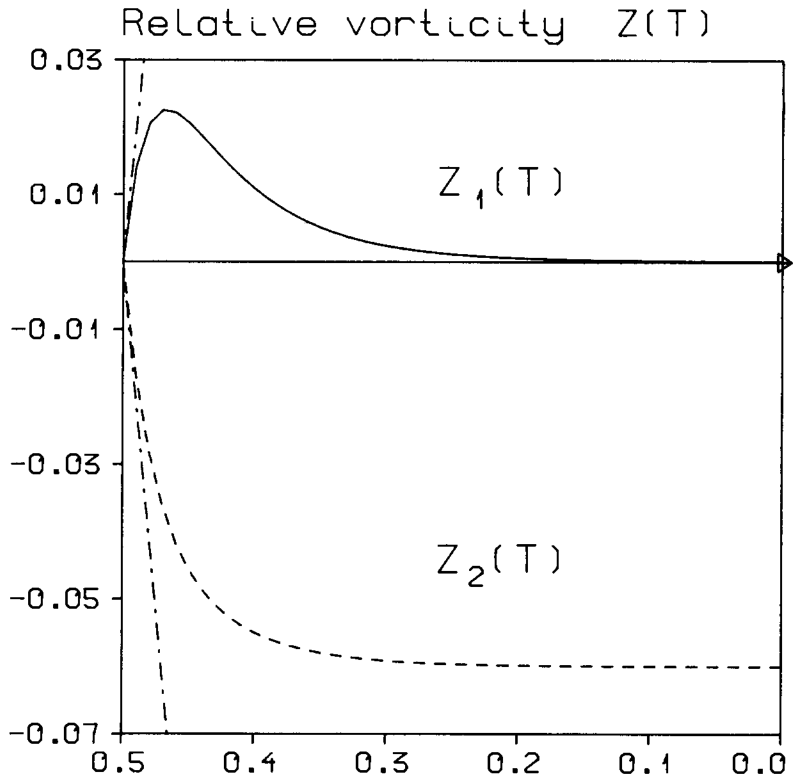

Figure 3. For legend see Figure 2. parameter values are Ro = 0.04, m = 0.25 and k = 0.5.

cases, it is readily deduced that the response is much slower (note that the vertical scale is not the same). The final (steady) states are only reached at the end of the time interval, $T \approx 0.0$, as opposed to the previous cases.

In concluding this section it may be appropriate to note that in all our computations the constant of integration A was at least two orders of magnitude less than the corresponding steady solutions. This is of interest in relation to the comments given at the end of the last section, cf. equations (22a,b).

5. The velocity components

In this section we will present the general expressions for the interior velocity components (U, V, W) in terms of the vorticity Z. The vertical velocity can be deduced from equation (1a), viz.

$$W_z = \frac{Z_t}{2\Omega}. \tag{24a}$$

This equation is readily integrated, and the result is given by

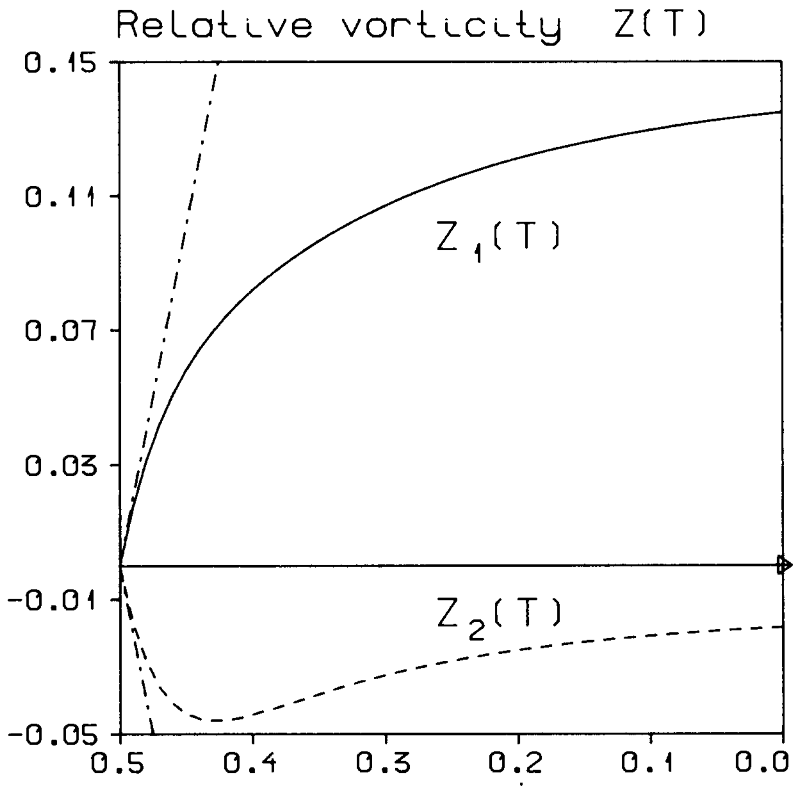

Figure 4. For legend see Figure 2. Parameter values are Ro = 0.04, $m = 0.9$ and $k = 3.0$.

$$W_1 = Z_{1t}(z-H)/2\Omega, \tag{24b}$$

$$W_2 = Z_{2t}z/2\Omega + q\frac{m}{k}\,\mathrm{Ro}^{-1}Z_2/2\Omega. \tag{24c}$$

The radial velocity then follows from the equation of continuity

$$(rU)_r = -rW_z = -rZ_t/2\Omega, \tag{25a}$$

subject to the boundary condition $U(b,t) = 0$. The solution is

$$U_1(r,t) = \frac{Z_{1t}}{4r\Omega}(b^2 - r^2), \tag{25b}$$

$$U_2(r,t) = \frac{Z_{2t}}{4r\Omega}(b^2 - r^2). \tag{25c}$$

It is possible to introduce a stream function $S = S(r,z,t)$ so that $rU = -S_z$ and $rW = +S_r$. A stream function, satisfying the condition $S(b,z,t) = \text{constant}$, is then given by

$$S_1 = Z_{1t}(b^2 - r^2)(H - z)/4\Omega, \tag{26a}$$

$$S_2 = -\left(zZ_{2t} + q\frac{m}{k}\,\mathrm{Ro}^{-1}Z_2\right)(b^2 - r^2)/4\Omega. \tag{26b}$$

At the initial instant $t = 0$, we have the relations

$$Z_{1t}(0) = +2q\Omega/H_1, \qquad Z_{2t}(0) = -2q\Omega/H_2, \tag{27a,b}$$

$$S_1(r,z,0) = \frac{q}{2}(b^2 - r^2)(H - z)/H_1, \tag{28a}$$

$$S_2(r,z,0) = \frac{q}{2}(b^2 - r^2)z/H_2. \tag{28b}$$

We note that the flow represented by (28a,b) is independent of the physical properties of the fluids, i.e. the parameters m and k. This is not very surprising, due to the constraint of conservation of the total fluid volume. This situation will be modified by viscous effects as the time increases, and the physical properties of the fluids will determine the development.

Let us turn our attention to the azimuthal velocity $V(r,t)$. The equation of motion is readily solved, viz.

$$V_t = -2\Omega U = -Z_t(b^2 - r^2)/2r, \tag{29a}$$

$$V(r,t) = -Z(t)(b^2 - r^2)/2r, \tag{29b}$$

where we omit the subscripts 1 and 2 for reasons of convenience.

The steady, azimuthal velocity in each layer is readily found from the steady value of the relative vorticity, cf. equations (7a,b). The steady, radial velocity is equal to zero in each layer, while the steady, vertical velocity is given by

$$W_{1s} = 0, \qquad W_{2s} = q(m-1). \tag{30a, b}$$

Since $m \leqslant 1$, it follows that $W_{2s} \leqslant 0$ when $q > 0$, and vice versa. The vertical velocity is the Ekman-suction into the Ekman-layer at the bottom of the annulus. It follows that if $m = 1$, there is no Ekman-suction and the vorticity must be zero.

In order to complete the solution in terms of the new time variable T, we will rewrite the stream function in the form $S = S(r,z,T)$. The result is given here by

$$S_1 = \frac{q}{2}(b^2 - r^2)\frac{H-z}{H_1}\frac{1-T_0}{1-T}\{1 + C_1(\check{Z}_2 - \check{Z}_1)\}, \tag{31a}$$

$$S_2 = \frac{q}{2}(b^2 - r^2)\frac{z}{H_2}\frac{T_0}{T}\{1 + C_2((1+2k)\check{Z}_2 - k\check{Z}_1)\} - \frac{q}{2}(b^2 - r^2)(1+k)C_2\check{Z}_2. \tag{31b}$$

The actual flow pattern represented by the stream function and the azimuthal velocity can readily be depicted, since the vorticity has been found. Since the vorticity is not in general a monotonic function of time (T), the radial velocity may very well change sign as time increases. The same behavior may be expected for the azimuthal velocity, cf. equations (25, 29).

6. Final remarks

In this paper we have studied the spin-up of a source-sink driven flow in a two-layer, rotating fluid. The governing vorticity equations were solved under the simplifying assumption of vanishing small internal and external Froude numbers. This is equivalent to the assumption of a flat interface and a flat upper surface, so we do not account for the vorticity induced by a (radial) flow across constant-depth contours, nor the vorticity induced by the time-dependent deviation from the moving interface and the upper surface. However, we are then able to solve the present problem in terms of well-known functions of mathematical physics (hypergeometric functions).

In Section 4 of this paper we have presented some results of our calculations. We have not considered such extreme cases as $m \ll 1$, $k \ll 1$ or $k \gg 1$, although such cases may be of some interest in laboratory experiments. We have given results which we believe represent the most important features of the spin-up process.

In conclusion, we also note that the one-layer sink flow, studied by Kranenburg [4], can be deduced from the present study if we (formally) set $m = k = 0$ and let $C_1 \rightarrow \infty$. A solution of equation (3a) is then $Z_1 = Z_2$. The remaining equation (3b) is then identical to the corresponding equation given in [4]. The present model may thus be more applicable to the situations studied by Kranenburg. It is clear that such important effects as mass entrainment at the interface and (air) stress at the upper free surface should be taken into account, so further studies are needed.

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